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Local models for smooth toral actions and residues

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For Stephen Halperin on his 50th birthday

Abstract

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If G is a compact Lie group acting smoothly on a manifold M, we prove that a G-invariant neighbourhood of the singular set Σ in M is completely determined by the G-vector bundle restriction of the tangent bundle of M to Σ. Moreover, by using only this G-vector bundle we define a *residual linear map*, above certain degrees, giving the Chern–Weil homomorphism for M, after composing it in cohomology with the inclusion of M in $(M, M - \Sigma)$. In the case of G being a torus, we characterize those G-vector bundles appearing as restriction to the singular set of the tangent bundle of some smooth G-manifold.

1. Introduction

When a compact Lie group acts on a manifold M, some of the topological properties of M are determined by those of the singular set Σ associate to the action or by the behaviour of the action near Σ. For instance, if M is compact, the Euler–Poincaré characteristics of M and Σ coincide.

In this paper G is a compact Lie group of dimension r acting smoothly on a smooth manifold M of dimension n and the singular set Σ is the set of those points $x \in M$ whose isotropy subgroups G_x have dimension at least one. Therefore the action of G is almost free on the complement of Σ in M.

The first theorem we prove, Theorem 1 in Section 2, says that a G-invariant neighbourhood of Σ in M is completly determined by the G-vector bundle $\eta = \tau_M|_\Sigma$, i.e. the restriction to Σ of the tangent bundle τ_M. On the other hand, it is well known

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that any characteristic class of M of degree greater than $n - r$ belongs to the image of the natural map $j: H^*(M, M - \Sigma) \to H^*(M)$, where the cohomology is taken with real coefficients. This suggests our definition, in Section 3, of a *residual linear map*, $\mathrm{Res}: \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \to H^{2p}(M, M - \Sigma)$, for $2p > n - r$ so that the composite $j \circ \mathrm{Res}$ is the Chern–Weil homomorphism for M. This *residual map* Res only depends on the G-bundle η, since for its construction we only need a G-neighbourhood of Σ, which is determined by η. In case M is compact and oriented we can use the Alexander–Lefschetz duality isomorphism, $T: H_{n-2p}(\Sigma) \to H^{2p}(M, M - \Sigma)$, to obtain a map $T^{-1} \circ \mathrm{Res}: \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \to H_{n-2p}(\Sigma)$.

Another description of the Alexander–Lefschetz duality, using a *Čech–de Rham* complex, for Σ being a stratified subset of M, is given by Lehmann in [5]. Check also [6], where he uses a particular case of this complex to obtain residues for singular foliations.

If G is a torus we characterize in this paper the G-vector bundles $\eta: E \xrightarrow{\pi} \Sigma$, appearing as restriction to its singular set Σ, of the tangent bundle of some smooth G-manifold. Thus we consider, in Section 4, a topological space Σ, a vector bundle η on Σ of rank n, which plays the role of $\tau_M|_\Sigma$, and a torus G of dimension r acting on η. Then we construct, under some natural restrictions on (Σ, η, G), a smooth manifold M of dimension n and a smooth action of G on M, such that the singular set of the action of G on M is Σ and η coincides with $\tau_M|_\Sigma$, see Theorem 2 in Section 5. Therefore, we obtain that (Σ, η, G) is the *local model* for the singular set of the action of a torus G on a smooth manifold. Furthermore, if Σ is compact, the manifold M can be chosen to be compact. This manifold M is not unique but, for instance, its Euler–Poincaré characteristic and, if M is compact and oriented, all its Pontrjagin numbers are completly determined.

On the other hand, we can construct a certain chain complex over the reals, using only the inner structure of the G-space Σ such that, if $H'_*(\Sigma)$ denotes the corresponding homology, one has a *generalized Thom isomorphism*, $T: H'_*(\Sigma) \to H^{n-*}(M, M - \Sigma)$. In case M is compact and oriented $H'_*(\Sigma)$ is isomorphic to the singular homology of Σ with real coefficients and T is Alexander–Lefschetz duality, see [4]. This allows us to regard in Section 6 the residual homomorphism associate to η as a linear map $\mathrm{Res}: \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \to H'_{n-2p}(\Sigma)$, such that for any smooth n-dimensional manifold M admitting a smooth action of G, with singular set Σ and $\eta = \tau_{M|\Sigma}$, the following diagram commutes for $2p > n - r$:

$$\begin{tikzcd} \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \arrow[r, "\mathrm{C.W.}"] \arrow[d, "\mathrm{Res}"] & H^{2p}(M) \\ H'_{n-2p}(\Sigma) \arrow[r, "T"] & H^{2p}(M, M - \Sigma). \arrow[u] \end{tikzcd}$$

Here C.W. denotes the Chern–Weil homomorphism.

Finally in Section 7 we give an example to illustrate all the above constructions.

2. The general case

Let G be a compact Lie group acting smoothly on the smooth manifolds M, M', with singular sets Σ, Σ', respectively.

Assume that we have a G-equivariant vector bundle isomorphism $(\varphi, \tilde{\varphi}) : \tau_M|_\Sigma \to \tau_{M'}|_{\Sigma'}$, where φ, $\tilde{\varphi}$ denotes respectively the maps for the bases or total spaces.

In particular, for each subgroup H of G with $\dim(H) \geq 1$, appearing as isotropy subgroup for the action of G, the restriction of φ to $F_H(M)$ is a G-equivariant homeomorphism onto $F_H(M')$, where $F_H(M)$, $F_H(M')$ denote the fixed point sets for the action of H on M, M', respectively.

We further assume that $\varphi : F_H(M) \to F_H(M')$ is smooth and $d\varphi$ coincides with the restriction of $\tilde{\varphi}$ to the tangent bundle of $F_H(M)$.

We then have the following theorem.

Theorem 1. *There exist G-invariant open neighbourhoods U, U' of Σ, Σ' in M, M' and a G-equivariant diffeomorphism $\bar{\varphi} : U \to U'$ extending φ, i.e. $\bar{\varphi}$, $d(\bar{\varphi})$ coincides with φ, $\tilde{\varphi}$ on $\tau_M|_\Sigma$.*

Proof. (a) It is enough to obtain a G-equivariant smooth map $\bar{\varphi} : U \to M'$ such that $\bar{\varphi}|_\Sigma = \varphi$ and $d(\bar{\varphi}) = \tilde{\varphi}$ on $\tau_M|_\Sigma$. In fact, $\bar{\varphi}$ is then a local diffeomorphism for the points of Σ and so it is a homeomorphism on some open neighbourhood of Σ in M, see Lemma 5.7 of [8]. Therefore it is a G-equivariant diffeomorphism on some G-invariant open neighbourhood of Σ in M.

(b) It is easy the construction of a $\bar{\varphi}$ as above in the neighbourhood of any point $x \in \Sigma$ by considering the Koszul model for the neighbourhood of the orbit of x.

(c) Suppose now that M' has finite orbit type, i.e. the number of conjugacy classes of isotropy subgroups is finite. Let V, U be G-invariant open subsets of M such that $\bar{V} \subset U$, and assume that $\psi : U \to M'$ is a smooth G-equivariant map such that coincides with φ on $U \cap \Sigma$ and $d\psi$ coincides with $\tilde{\varphi}$ on the restriction of τ_M to $U \cap \Sigma$. We then have the following lemma.

Lemma. *There exist a G-invariant open neighbourhood W of Σ in M such that $\bar{V} \subset W$ and a G-equivariant smooth map $\bar{\varphi} : W \to M'$ such that $\bar{\varphi}$ coincides with ψ on $\bar{V}$, $\bar{\varphi}|_\Sigma = \varphi|_\Sigma$ and $d(\bar{\varphi})$ coincides with $\tilde{\varphi}$ on the restriction of τ_M to Σ.*

Proof. To prove this lemma observe that, since M' has finite orbit type, we may consider M' as a G-invariant closed submanifold of some Euclidean space $\mathbb{R}^N$ endowed with a linear representation of G. We choose also a G-invariant tubular neighbourhood $\rho' : T' \to M'$ of M' in $\mathbb{R}^N$. Then, by using (b), we can find G-invariant open sets U_i, such that $U_i \cap \bar{V} = \emptyset$, covering $M - \bar{V}$, and G-equivariant smooth maps $\varphi_i : U_i \to M'$ such that φ_i coincides with φ on $\Sigma \cap U_i$ and $d\varphi_i$ coincides with $\tilde{\varphi}$ on the restriction of τ_M to $\Sigma \cap U_i$. Set now $f = \lambda \cdot \psi + \sum \lambda_i \cdot \varphi_i$, where λ, λ_i is a G-invariant partition of unity subordinate to the open covering $\{U, U_i\}$. Then $W = f^{-1}(T')$, and $\bar{\varphi} = \rho' \circ f$ satisfy the conditions of the lemma. □

(d) To finish the proof of Theorem 1, choose now a sequence of G-invariant open sets in M', $U'_1, U'_2, \ldots$ such that the closures $\overline{U'_i}$ are compacts, $\overline{U'_i} \subset U'_{i+1}$, for all $i \geq 1$ and $\bigcup_{i \geq 1} U'_i \supset \Sigma'$. There exists then a sequence of G-invariant open sets in M, $U_1, U_2, \ldots$ such that the closures $\overline{U_i}$ are compacts, $\overline{U_i} \subset U_{i+1}$ and $\varphi(U_i \cup \Sigma) = U'_i \cup \Sigma'$, for all $i \geq 1$. In particular $\bigcup_{i \geq 1} U_i \supset \Sigma$.

Observe that the action of G on U' has finite orbit type, due to the fact the $\overline{U'_i}$ is compact. Thus an obvious inductive argument using the lemma above allows the construction of the required $\bar{\varphi}$. □

Remarks. (a) Observe that Theorem 1 holds without any restriction on the number of orbit types for the actions of G.

(b) $H^*(M, M - \Sigma)$, with real coefficients, only depends on the G-vector bundle $\tau_{M|\Sigma}$. Of course, this is clear if M is compact and oriented, since Σ is locally contractible, and so we have Alexander–Lefschetz duality isomorphism $T: H_{n-q}(\Sigma) \to H^q(M, M - \Sigma)$.

3. The residual map

Let G be a compact Lie group of dimension r acting smoothly on a smooth manifold M of dimension n, and let Σ be its singular set.

For each G-open neighbourhood U of Σ in M, we denote by $H^*_{\text{f.c.}}(U)$ the cohomology of the subcomplex of de Rham forms on M whose supports lie in U. One checks easily that integration gives a natural isomorphism with $\varinjlim_{C \subset U} H^*(M, M - C)$, for real coefficients, where C runs through the closed subsets of M contained in U.

Next, we use the Chern–Weil construction with G-invariant linear connections ∇ such that ∇_{Z_h} equals the Lie derivative with respect to the fundamental vector field Z_h, for any h in the Lie algebra of G, outside of some closed subset C of M, $C \subset U$. In this way we obtain a well-defined linear map $w_U: \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \to H^{2p}_{\text{f.c.}}(U)$, for $2p > n - r$. But, if $U \subset V$ are G-invariant open neighbourhoods of Σ, w_V coincides with the composite of w_U and the canonical map $H^{2p}_{\text{f.c.}}(U) \to H^{2p}_{\text{f.c.}}(V)$. Therefore, we have a linear map

$$\mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \to \varprojlim_{U \supset \Sigma} H^{2p}_{\text{f.c.}}(U).$$

Finally if we compose the above map with the isomorphisms

$$\varprojlim_{U \supset \Sigma} H^{2p}_{\text{f.c.}}(U) \to \varprojlim_{U \supset \Sigma} \big(\varinjlim_{C \subset U} H^{2p}(M, M - C)\big) \leftarrow H^{2p}(M, M - \Sigma),$$

we obtain, by definition, the *residual map*

$$\mathrm{Res}: \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \to H^{2p}(M, M - \Sigma), \quad 2p > n - r.$$

It is clear that the Chern–Weil homomorphism for $2p > n - r$,

$$\mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \to H^{2p}(M),$$

is the composite of Res and the natural map $H^{2p}(M, M - \Sigma) \to H^{2p}(M)$.

Observe that the residual map Res only depends on the G-vector bundle $\eta = \tau_M|_\Sigma$. In fact, by using Theorem 1 we obtain a commutative diagram

$$\begin{tikzcd} & \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \arrow[dl, "\mathrm{Res}"'] \arrow[dr, "\mathrm{Res}"] & \\ H^{2p}(M, M - \Sigma) \arrow[rr, "\cong"] & & H^{2p}(M', M' - \Sigma') \end{tikzcd}$$

if G acts on M, M' with $\tau_M|_\Sigma \cong \tau_{M'}|_{\Sigma'}$ as in Section 2.

In particular observe that the characteristic classes of degree greater than $n - r$ of a smooth manifold M, acted on by a compact Lie group G of dimension r, only depend on the restriction of the tangent bundle to its singular set and the action of G on that bundle.

4. The local model for a toral action

In this section we introduce some notation and specify the hypotheses we need for the validity of Theorem 2. Our purpose is to describe the G-vector bundle $\tau_M|_\Sigma$ without any mention of the manifold M, for the particular case of G being a torus.

Our local model for a toral action at the singular set consists of an oriented G-vector bundle of rank n, $\eta: E \xrightarrow{\pi} \Sigma$, having as base a Hausdorff, second countable space Σ. We assume that G is an r-dimensional torus acting continuously from the left both on E and Σ so that π is equivariant, G preserves the orientation, the action of G on E is effective, i.e. the unit element of G is the only one fixing all points in E, and the action of G on Σ is singular, i.e. there is no orbit of dimension r.

Let $\mathscr{G}$ be the family of subtori appearing as 1-component of isotropy subgroups for the action of G on Σ. If $H \in \mathscr{G}$, F_H denotes the fixed point set for the induced action of H on Σ. It is a closed subset of Σ with all its connected components G-invariant.

Let $\mathscr{F}$ be the family of all connected components of F_H for all $H \in \mathscr{G}$. It is clear that $\mathscr{F}$ is a convering of Σ and the intersection of any two members of $\mathscr{F}$ is union of members of $\mathscr{F}$.

Let $S(F)$ be the maximum subtorus in $\mathscr{G}$ fixing F, for each $F \in \mathscr{F}$. We assume that if the isotropy subgroup G_z, for $z \in E$, is not finite, there exists then $F \in \mathscr{F}$ such that $\pi(z) \in F$ and $G_z \supset S(F)$.

We suppose that the restriction η_F of the bundle η to each $F \in \mathscr{F}$ is a smooth G-vector bundle, i.e. we suppose that F and $\pi^{-1}(F)$ are smooth manifolds such that π and the action of G on η_F are smooth, and η_F has a smooth trivializing system. We further assume that whenever $F \subset F'$, F, F' in $\mathscr{F}$, then F is a smooth submanifold of F'. In particular, $\pi^{-1}(F)$ is also a smooth submanifold of $\pi^{-1}(F')$.

Observe that $S(F)$ induces representations on the fibres of η_F. Define then $\varphi_F(z) = \int_{S(F)} az\, da$, for each $z \in \pi^{-1}(F)$. We obtain clearly a G-equivariant smooth endomorphism of η_F such that $\varphi_F(az) = \varphi_F(z)$ for all $a \in S(F)$ and $\varphi_F^2 = \varphi$. In particular, φ_F has constant rank, the kernel and image of φ_F are smooth subbundles of η_F and we have the canonical G-invariant decomposition $\eta_F = \operatorname{Ker} \varphi_F \oplus \operatorname{Im} \varphi_F$ with $\operatorname{Ker} \varphi_F = \{z \in \pi^{-1}(F) \mid az = z \text{ for all } a \in S(F)\}$.

Finally we assume that $\operatorname{Ker} \varphi_F = \tau_F$, the tangent bundle of F, and the action of G on τ_F is induced by the action of G on F. We write $\nu_F = \operatorname{Im} \varphi_F$ and so we have canonical G-invariant decompositions $\eta_F = \tau_F \oplus \nu_F$, for each $F \in \mathscr{F}$.

The representations of $S(F)$ on the fibres of ν_F are equivalent since they all have the same character, namely rank φ_F. Therefore, the canonical decomposition of the representation of $S(F)$ on each fibre induces a canonical decomposition $\nu_F = \nu_F^1 \oplus \cdots \oplus \nu_F^{t(F)}$, where each ν_F^k is a G-invariant vector bundle. We also set $\nu_F^0 = \tau_F$ and so we can write $\eta_F = \bigoplus_{k=0}^{t(F)} \nu_F^k$.

If $F \subset F'$, F, F' in $\mathscr{F}$, we have $S(F) \supset S(F')$ and the restriction of $\nu_{F'}^k$ to F, $k = 0, \ldots, t(F')$, is a sum of some of the bundles ν_F^j. In particular, $\tau_{F'}|_F = \tau_F \oplus \nu_{FF'}$, where $\nu_{FF'}$ is a sum of some bundles ν_F^k.

Observe that we can define complex structures on each ν_F^k, $k \neq 0$, such that the representation of $S(F)$ on ν_F^k is given by $av = \lambda_F^k(a)v$, for a Lie group epimorphism $\lambda_F^k : S(F) \to S^1$, and the action of G on ν_F^k is complex.

The above complex structures are unique up to conjugation and clearly can be chosen in a compatible way, i.e. if $F \subset F'$, F, F' in $\mathscr{F}$, and $\nu_{F'}^k|_F = \bigoplus_{j \in J} \nu_F^k$ for some subset $J \subset \{1, \ldots, t(F)\}$, then $\lambda_{F'}^k$ is the restriction of λ_F^j to $S(F')$ and the complex structures on $\nu_{F'}^k|_F$ is the sum of the complex structures on the bundles ν_F^j, $j \in J$.

Remarks. (a) $t(F) \geq \dim S(F)$ for each $F \in \mathscr{F}$.

(b) rank $\nu_F = n - \dim F$ and the dimension of the total space of ν_F is n for all $F \in \mathscr{F}$.

(c) $\sum_{F' \supset F} \tau_{F'}|_F$ is a direct sum of τ_F and some ν_F^k. Therefore, it is a vector bundle over F. This bundle coincides with η_F if $\dim S(F) \geq 2$ and coincides with τ_F if $\dim S(F) = 1$. In fact, choose $x \in F$ with $G_x^0 = S(F)$ and let z be an element of the total space of ν_F^k, $k \neq 0$, such that $\pi(z) = x$. If $\dim S(F) \geq 2$, then $\dim G_z \geq 1$ and so, by hypothesis, there exists $F' \in \mathscr{F}$ such that $x \in F'$ and $G_z \supset S(F')$. Therefore, $z \in T_x(F')$ and we have $F' \supset F$, since $x \in F \cap F'$ and $S(F') \subset S(F) = G_x^0$. If, instead, $\dim S(F) = 1$, then G_z is finite and so $z \notin T_x(F')$ for any $F' \supset F$.

Example. The following example illustrate the local model we have just described.

Let Σ be the topological space obtained from three copies of $\mathbb{C}P^1$, $\mathbb{C}P^1 \times \{0, 1, 2\}$, by the identifications

$$A = (\langle 1, 0\rangle, 0) = (\langle 0, 1\rangle, 2), \quad B = (\langle 0, 1\rangle, 0) = (\langle 0, 1\rangle, 1),$$
$$C = (\langle 1, 0\rangle, 1) = (\langle 1, 0\rangle, 2).$$

Let G be the 2-dimensional torus $S^1 \times S^1$ and consider the action on $\mathbb{C}P^1 \times \{0, 1, 2\}$ given by

$$(a, b) \cdot (\langle z, z'\rangle, 0) = (\langle az, bz'\rangle, 0),$$
$$(a, b) \cdot (\langle z, z'\rangle, 1) = (\langle z, bz'\rangle, 1),$$
$$(a, b) \cdot (\langle z, z'\rangle, 2) = (\langle z, az'\rangle, 2).$$

It is clear that the above action induces one on Σ with $\mathcal{G} = \{G, S^1 \times 1, 1 \times S^1, S\}$, where $S = \{(a, a)\}_{a \in S^1}$, $F_G = \{A, B, C\}$, $F_{S^1 \times 1} = \mathbb{C}P^1 \times 1 \cup \{A\}$, $F_{1 \times S^1} = \mathbb{C}P^1 \times 2 \cup \{B\}$ and $F_S = \mathbb{C}P^1 \times 0 \cup \{C\}$.

Therefore, $\mathcal{F} = \{A, B, C, \mathbb{C}P^1 \times 0, \mathbb{C}P^1 \times 1, \mathbb{C}P^1 \times 2\}$, $S(A) = S(B) = S(C) = G$, $S(\mathbb{C}P^1 \times 0) = S$, $S(\mathbb{C}P^1 \times 1) = S^1 \times 1$ and $S(\mathbb{C}P^1 \times 2) = 1 \times S^1$.

Let $\tau : T \to \mathbb{C}P^1$ be the tangent bundle of $\mathbb{C}P^1$ and $v' : L' \to \mathbb{C}P^1$ the dual of the canonical complex line bundle, $L' = \{(\langle z, z'\rangle, (\lambda, \mu)) \in \mathbb{C}P^1 \times \mathbb{C}^2 \mid \mu\bar{z} = \lambda\bar{z}'\}$, and let $\tilde{E} \to \mathbb{C}P^1$ be the direct sum of τ and v'.

Endow $T \times \{0, 1, 2\}$ with the action induced from the action of G on $\mathbb{C}P^1 \times \{0, 1, 2\}$ and $L' \times \{0, 1, 2\}$ with the action given by

$$(a, b) \cdot (\langle z, z'\rangle, (\lambda, \mu)) \times 0 = (\langle az, bz'\rangle, (a^{-1}\lambda, b^{-1}\mu)) \times 0,$$
$$(a, b) \cdot (\langle z, z'\rangle, (\lambda, \mu)) \times 1 = (\langle z, bz'\rangle, (a\lambda, ab^{-1}\mu)) \times 1,$$
$$(a, b) \cdot (\langle z, z'\rangle, (\lambda, \mu)) \times 2 = (\langle z, az'\rangle, (b\lambda, a^{-1}b\mu)) \times 2.$$

Let $\eta : E \xrightarrow{\pi} \Sigma$ be the G-vector bundle, where E is the space obtained from $\tilde{E} \times \{0, 1, 2\}$ by identifying according to the following G-equivariant isomorphisms of complex vector spaces, and π is the obvious projection.

$$\tilde{E}_{\langle 1,0\rangle} \times 0 = (T_{\langle 1,0\rangle} \oplus L'_{\langle 1,0\rangle}) \times 0 \to \tilde{E}_{\langle 0,1\rangle} \times 2 = (T_{\langle 0,1\rangle} \oplus L'_{\langle 0,1\rangle}) \times 2,$$
$$(\lambda X\langle 1, 0\rangle, (\langle 1, 0\rangle, (\mu, 0))) \times 0 \mapsto (\mu Y\langle 0, 1\rangle, (\langle 0, 1\rangle, (0, \lambda))) \times 2.$$

$$\tilde{E}_{\langle 0,1\rangle} \times 0 = (T_{\langle 0,1\rangle} \oplus L'_{\langle 0,1\rangle}) \times 0 \to \tilde{E}_{\langle 0,1\rangle} \times 1 = (T_{\langle 0,1\rangle} \oplus L'_{\langle 0,1\rangle}) \times 1,$$
$$(\lambda Y\langle 0, 1\rangle, (\langle 0, 1\rangle, (0, \mu))) \times 0 \mapsto (\mu Y\langle 0, 1\rangle, (\langle 0, 1\rangle, (0, \lambda))) \times 1.$$

$$\tilde{E}_{\langle 1,0\rangle} \times 1 = (T_{\langle 1,0\rangle} \oplus L'_{\langle 1,0\rangle}) \times 1 \to \tilde{E}_{\langle 1,0\rangle} \times 2 = (T_{\langle 1,0\rangle} \oplus L'_{\langle 1,0\rangle}) \times 2,$$
$$(\lambda X\langle 1, 0\rangle, (\langle 1, 0\rangle, (\mu, 0))) \times 1 \mapsto (\mu X\langle 1, 0\rangle, (\langle 1, 0\rangle, (\lambda, 0))) \times 2.$$

Here X denotes the vector field $\partial/\partial x$ on $\mathbb{C}P^1 - \langle 0, 1\rangle$ corresponding to the local coordinates $\psi : \mathbb{C}P^1 - \langle 0, 1\rangle \to \mathbb{C}$ given by $\psi(\langle z, z'\rangle) = z'/z$, and Y is the vector field $\partial/\partial x$ on $\mathbb{C}P^1 - \langle 1, 0\rangle$ corresponding to the local coordinates $\psi' : \mathbb{C}P^1 - \langle 1, 0\rangle \to \mathbb{C}$ given by $\psi'(\langle z, z'\rangle) = z/z'$.

The G-vector bundle $\eta : E \xrightarrow{\pi} \Sigma$ of this example clearly satisfies all our hypotheses and so it is a *local model* for a 2-dimensional toral action. Of course, in this case one can easily check that η is isomorphic to the G-bundle obtained by restricting the tangent bundle of $\mathbb{C}P^2$ to its singular set for the action given by $(a, b) \cdot \langle z_0, z_1, z_2\rangle = \langle z_0, az_1, bz_2\rangle$.

5. Existence of a G-manifold with the given model

Let $\eta : E \xrightarrow{\pi} \Sigma$ be a G-bundle satisfying the hypotheses of Section 4. Theorem 2 then reads as follows:

Theorem 2. *There exists a smooth manifold M, together with a smooth action of G, such that η is G-equivariantly isomorphic to the restriction of the tangent bundle of M to the singular set of the action of G on M. Furthermore, if Σ is compact, M can be chosen to be compact.*

We need first the following technical theorem, where $\pi_F : N_F \to F$, respectively $\pi_F : N_{FF'} \to F$, denotes the projection, total space and base of ν_F, respectively $\nu_{FF'}$, for $F \in \mathscr{F}$, respectively $F \subset F'$ in $\mathscr{F}$.

A technical theorem. *There exist G-equivariant smooth maps $\psi_{FF'} : N_F \to N_{F'}$, $F \subset F'$, F and F' in $\mathscr{F}$, such that each $\psi_{FF'}$ is a diffeomorphism onto its image, is the identity on $\pi_{F'}^{-1}(F) = N_F \cap N_{F'}$, and satisfies the following four properties.*

(i) $\psi_{F'F''} \circ \psi_{FF'}$ *whenever* $F \subset F' \subset F''$, F, F', F'' *in* $\mathscr{F}$.

Observe that $\psi_{FF'}(N_{FF'}) = \psi_{FF'}(N_F) \cap F'$, since $N_{FF'}$ is the submanifold fixed by $S(F')$ in $N_{F'}$, here F' is identified with its image under the zero-cross section in $N_{F'}$. Set $U_{FF'} = \psi_{FF'}(N_F) \cap F'$ and $\rho_{FF'} = \pi_F \circ \psi_{FF'}^{-1} : U_{FF'} \to F$. It is a G-tubular neighbourhood of F in F'.

(ii) $\pi_{F'}^{-1}(U_{FF'}) = \psi_{FF'}(N_F)$, *and we have a vector bundle isomorphism*

$$\begin{tikzcd}
N_F \arrow[r, "\psi_{FF'}"] \arrow[d, "\pi_{FF'}"] & \pi_{F'}^{-1}(U_{FF'}) \arrow[r] \arrow[d, "\pi_{F'}"] & N_{F'} \arrow[d, "\pi_{F'}"] \\
N_{FF'} \arrow[r, "\psi_{FF'}"] & U_{FF'} \arrow[r] & F'
\end{tikzcd}$$

(iii) *If $F \subset F''$, $F' \subset F''$, $F \cap F' = \emptyset$, then $U_{FF''} \cap U_{F'F''} = \emptyset$, for F, F', F'' in $\mathscr{F}$.*

(iv) *If $F_1 \subset F_3$ and $F_2 \subset F_3$, F_1, F_2, F_3 in $\mathscr{F}$, then the family of U_{FF_3}, respectively of $\bar{U}_{FF_3}$, for each connected component F of $F_1 \cap F_2$, coincides with the family of connected components of $U_{F_1F_3} \cap U_{F_2F_3}$, respectively of $\bar{U}_{F_1F_3} \cap \bar{U}_{F_2F_3}$.*

Remarks. (a) The next two properties are obtained easily from those above.

(v) If $F \subset F' \subset F''$, F, F', F'' in $\mathscr{F}$, then

$$\rho_{F'F''}^{-1}(U_{FF'}) = U_{FF''} \quad \text{and} \quad \rho_{FF'} \circ \rho_{F'F''}|_{U_{FF''}} = \rho_{FF''}.$$

(vi) If $F_1 \subset F_3$, $F_2 \subset F_3$ and F is a component of $F_1 \cap F_2$, F, F_1, F_2, F_3 in $\mathscr{F}$, then U_{FF_1} is the component of $F_1 \cap U_{F_2F_3}$ containing F and $\rho_{FF_1} = \rho_{F_2F_3}|_{U_{FF_1}}$.

(b) The family $\mathscr{T} = \{(U_{FF'}, \rho_{FF'})\}_{F \subset F'}$ was called in [4] an internal tubular neighbourhood for $\mathscr{F}$.

Proof of Theorem 2. Let M be the space obtained from the disjoint union (see Fig. 1) of the N_F, for all $F \in \mathscr{F}$, by identifying N_F with its image in N'_F under the isomorphism $\psi_{FF'}$, whenever $F \subset F'$, F, F' in $\mathscr{F}$. M is then an n-dimensional smooth manifold which is covered by open sets diffeomorphic to N_F, for all $F \in \mathscr{F}$. The only point one has to check is that M is Hausdorff, but this can be observed easily by using the above theorem.

M inherits a G-action, since each $\psi_{FF'}$ is G-equivariant and so we have a canonical G-equivariant inclusion $\Sigma \hookrightarrow M$ such that each $F \in \mathscr{F}$ is a closed submanifold of M. It is clear that Σ coincides with the singular set of the action of G on M, since if $z \in M$, for instance $z \in N_F$, with $\dim G_z \geq 1$, there exists then $F' \in \mathscr{F}$ with $F' \supset F$ such that $z \in T_x(F')$ for $x \in F$. Therefore, $z \in N_{FF'}$ and so $\psi_{FF'}(z) \in F' \subset \Sigma$.

On the other hand, $\tau_M|_F = \tau_{N_F}|_F = \tau_F \oplus \nu_F = \eta_F$.

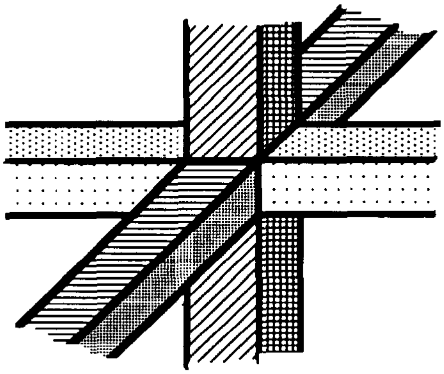

Fig. 1.

Finally, if Σ is compact, choose a nonnegative proper G-invariant Morse function $f: M \to \mathbb{R}$ and let c be a real number such that $\Sigma \subset M^c = \{x \in M \mid f(x) \leq c\}$. M^c is a compact, G-invariant, smooth submanifold of M with boundary ∂M^c such that $\partial M^c \cap \Sigma = \emptyset$; see [7].

Let M' be the double of M^c with the corresponding smooth G-action. Its singular set consists of two disjoint copies of Σ. We proceed then, by induction on the dimensions of the submanifolds $F \in \mathscr{F}$, to blow up G-equivariantly one copy of Σ, see [3], and so we end up with a compact smooth G-manifold N with singular set Σ and such that η is G-equivariantly isomorphic to the restriction of the tangent bundle of N to Σ. $\square$

Proof of the technical theorem. (a) Choose a family $\{W_F\}_{F \in \mathscr{F}}$ of G-invariant open sets of Σ such that $W_F \supset F$ for all $F \in \mathscr{F}$ and $\bar{W}_F \cap \bar{W}_{F'} = \emptyset$ if and only if $F \cap F' = \emptyset$, for F, F' in $\mathscr{F}$. This can be done easily since our hypotheses imply the normality of the orbit space Σ/G.

(b) For the construction below we need the following lemmas.

Lemma 3. *Let $\xi: E \xrightarrow{\pi} B$ be a smooth G-vector bundle, G a compact Lie group, and suppose $B \subset C \subset U \subset E$, where B is identified with its image under the zero-cross section, C is a G-invariant closed neighbourhood of B, U is a G-invariant open neighbourhood of B. Further assume that $C \cap \pi^{-1}(x)$ is compact and $U \cap \pi^{-1}(x)$ is starlike, for each $x \in B$. There exists then a G-invariant smooth map $\lambda: E \to \mathbb{R}^{>0}$ such that the restriction of λ to C is 1 and $q: E \to E$ given by $q(z) = \lambda(z) \cdot z$ is a G-equivariant diffeomorphism onto U.*

Proof. To prove this lemma, choose first a G-invariant Riemannian metric on ξ such that $C \subset D_1 \subset D_2 \subset U$, where D_1, D_2 denote the disc bundles of radii 1 and 2 respectively. Choose then a G-invariant smooth function $g: U \to \mathbb{R}$ such that g is zero on D_1 and g restricted to $U \cap \pi^{-1}(x)$ is proper for all $x \in B$. Observe that the vector field $X = e^{-(Zg)^2} \cdot Z$ is globally integrable on U, where Z denotes the *radial* vector field, $Z(z) = z$, $z \in U$. Let $\varphi: \mathbb{R} \times U \to U$ be the flow of X. Define $q: E \to U$ by

$$q(z) = \begin{cases} \varphi(\log|z|, z/|z|) & \text{if } |z| \geq 1, \\ z & \text{if } |z| < 1 \end{cases}$$

and let $\lambda: E \to \mathbb{R}^{>0}$ be the unique map such that $q(z) = \lambda(z) \cdot z$ for all $z \in E$. These q and λ satisfy the required conditions. $\square$

Lemma 4 (Extension lemma). *Let F be a member of $\mathscr{F}$ and suppose that V, U are G-invariant open subsets of F such that $\bar{V} \subset U$. Fix a G-equivariant map $h: \pi_F^{-1}(U) \to \Sigma$ such that its restriction to U is the identity and $h: \pi_F^{-1}(U) \cap N_{FF'} \to F'$ is smooth for all $F' \in \mathscr{F}$, $F' \supset F$. Assume that we are given smooth G-equivariant maps*

$\varphi_{FF'}: N_{FF'} \to F'$, for all $F' \in \mathscr{F}$, $F' \supset F$, such that $\varphi_{FF'}$ is the identity on F and $\varphi_{FF'}$ coincides with h on $N_{FF'} \cap \pi_F^{-1}(U) \cap W_0$ for some G-invariant open neighbourhood W_0 of F in N_F.

Then, there exists a G-invariant open neighbourhood W of F in N_F and a G-equivariant map $\varphi_F: N_F \to \Sigma$ such that $\varphi_F: N_{FF'} \to F'$ is smooth for all $F' \in \mathscr{F}$, $F' \supset F$, φ_F coincides with $\varphi_{FF'}$ on $W \cap N_{FF'}$ and φ_F coincides with h on $W \cap \pi_F^{-1}(V)$. □

This lemma is proved using standard techniques in differential topology.

(c) Using now the extension lemma above, together with Lemma 3 and [8, Lemma 5.7], as in part (a) of the proof of Theorem 1, we can show that a neighbourhood of any non-maximal member F of $\mathscr{F}$ looks like $\bigcup_{F' \supset F} N_{FF'} \subset N_F$. More precisely, there exists a G-invariant open neighbourhood $U_F \subset W_F$, of F in Σ, and a G-equivariant homeomorphism $\psi_F: \bigcup_{F' \supset F} N_{FF'} \to U_F$ such that $\psi_F: N_{FF'} \to U_{FF'} = U_F \cap F'$ is a diffeomorphism for all $F' \supset F$, $F' \in \mathscr{F}$. Besides we also construct G-equivariant vector bundle isomorphisms

$$\begin{CD}
N_F @>{\psi_{FF'}}>> \pi_{F'}^{-1}(U_{FF'}) @>>> N_{F'} \\
@V{\pi_{FF'}}VV @V{\pi_{F'}}VV @V{\pi_{F'}}VV \\
N_{FF'} @>{\psi_{FF'}}>> U_{FF'} @>>> F'
\end{CD}$$

This can be done by choosing first G-invariant tubular neighbourhoods $\psi_{FF'}: N_{FF'} \to U_{FF'}$ for the members F' of $\mathscr{F}$, such that $F \subset F'$, but no other member F'' of $\mathscr{F}$ satisfies $F \subset F'' \subset F'$, and then extending inductively to the other members of $\mathscr{F}$ containing F, by using the previous lemmas.

(d) Finally, a step by step procedure, using again the lemmas above, gives the required construction. □

Observe that Theorem 1 proves the local uniqueness of the previous construction.

6. The residual map for a toral action

Let $\eta: E \xrightarrow{\pi} \Sigma$ be a G-bundle satisfying the hypotheses of Section 4, for G a torus. We also keep the same notation.

We recall first the construction of a chain complex $(A'_*(\Sigma), d)$ associate to the G-space Σ, whose homology, $H'_*(\Sigma)$, is isomorphic to $H^{n-*}(M, M - \Sigma)$ by a generalized Thom isomorphism, whenever Σ is the singular set for the action of G on M and $\eta = \tau_M|_\Sigma$, see [4]. Set $A_p(\Sigma) = \prod_{F \in \mathscr{F}} A_p(F)$, where $A_p(F)$ are the de Rham forms of F, of degree $\dim(F) - p$, and $A_p(F) = 0$ for $p > \dim(F)$. $A'_p(\Sigma)$ is then the quotient of $A_p(\Sigma)$ by the subspace spanned by the elements of the form $(\Phi, -\int\Phi) \in A_p(F') \oplus A_p(F)$ for $F \subset F'$, F, F' in $\mathscr{F}$, where Φ has fibre compact support in $U_{FF'}$ and $\int$ denotes the corresponding fibre integral for the oriented G-invariant tubular neighbourhood $\rho_{FF'}: U_{FF'} \to F$.

Denote by $H'_*(\Sigma)$ the homology of the chain complex $(A'_*(\Sigma), d)$, where d is induced by the usual exterior derivative.

We proceed now as follows:

Choose a G-manifold M with a family of tubular neighbourhoods U_F as in Section 5, let U be the union of all U_F and so we have a linear map, for $2p > n - r$, $w_U: \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \to H^{2p}_{\mathrm{f.c.}}(U)$, given in Section 3 by using a Baum–Cheeger connection in the Chern–Weil construction so that the corresponding forms of degree greater than $n - r$ vanish outside a closed subset of M contained in U. We also have a generalized Thom isomorphism, see [4, Section 3] $T_U: H^{2p}_{\mathrm{f.c.}}(U) \to H'_{n-2p}(\Sigma)$ induced by the chain map $A^{2p}_{\mathrm{f.c.}}(U) \to A'_{n-2p}(\Sigma)$ that sends $\sum_{F \in \mathscr{F}} \Phi_F$, $\Phi_F \in A^{2p}_{\mathrm{f.c.}}(U)$, to the class of $(\int \Phi_F)_{F \in \mathscr{F}}$ in $A'_{n-2p}(\Sigma)$.

We can define then a residual map, $\mathrm{Res}': \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \to H'_{n-2p}(\Sigma)$ as the composition of w_U and the generalized Thom isomorphism $T_U: H^{2p}_{\mathrm{f.c.}}(U) \to H'_{n-2p}(\Sigma)$.

Theorem. *The residual map,* Res′, *defined above depends only on the G-bundle* η.

Proof. Let M and M' be two smooth G-manifolds containing Σ as its singular set and such that the restriction of their tangent bundles to Σ coincide with η. We know that we can choose a system of G-tubular neighbourhoods $\{U_F\}_{F \in \mathscr{F}}$, $\{U'_F\}_{F \in \mathscr{F}}$ and a G-equivariant diffeomorphism $f: U = \bigcup_F U_F \to U' = \bigcup_F U'_F$ such that $f(U_F) = U'_F$, for all $F \in \mathscr{F}$. The commutativity of the following diagram finishes the proof:

$$\begin{tikzcd} & \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \arrow[dl, "w_{U'}"'] \arrow[dr, "w_U"] & \\ H^{2p}_{\mathrm{f.c.}}(U') \arrow[rr, "f^*"] \arrow[dr, "T_{U'}"'] & & H^{2p}_{\mathrm{f.c.}}(U) \arrow[dl, "T_U"] \\ & H'_{n-2p}(\Sigma) & \end{tikzcd}$$

Finally, it is clear that the following diagram commutes where Res is the residual map defined in Section 3,

$$\begin{tikzcd} & \mathrm{Sym}^p(\mathrm{Sk}_{\mathbb{R}^n})_I \arrow[dl, "\mathrm{Res}"'] \arrow[dr, "\mathrm{Res}'"] & \\ H^{2p}(M, M - \Sigma) \arrow[rr, "\cong"] & & H'_{n-2p}(\Sigma) \end{tikzcd}$$

if G acts on M with $\eta \cong \tau_M|_\Sigma$. $\square$

Remark. The result above can be generalized easily for characteristic classes of principal bundles. Let η be as above and assume that $\mathscr{P}$ is a principal bundle with structure group K and base Σ, acted on by the torus G. We can obtain then a residual map $\mathrm{Res}:\mathrm{Sym}^p(\mathbf{K})_I \to H_{n-2p}(\Sigma)$, for $2p > n-r$, where $\mathrm{Sym}^p(\mathbf{K})_I$ denotes the algebra of ad-invariant multilinear symmetric functions in the Lie algebra $\mathbf{K}$ of K.

7. Example

For our example in Section 4, we can construct the manifold M in the following way.

Let $F_0 = \mathbb{C}P^1 \times 0$, $F_1 = \mathbb{C}P^1 \times 1$ and $F_2 = \mathbb{C}P^1 \times 2$. Then $\mathscr{F} = \{A, B, C, F_0, F_1, F_2\}$. We define

$$U_{AF_0} = \{\langle z_0, z_1\rangle \in \mathbb{C}P^1 \,|\, |z_1| < |z_0|\} \times 0,$$

$$U_{BF_0} = \{\langle z_0, z_1\rangle \in \mathbb{C}P^1 \,|\, |z_0| < |z_1|\} \times 0.$$

Observe that $U_{AF_0} \cap U_{BF_0} = \emptyset$ and $U_{AF_0} \cup U_{BF_0}$ is the complement in F_0 of the circle $\{\langle z_0, z_1\rangle \in \mathbb{C}P^1 \,|\, |z_0| = |z_1|\} \times 0$. Similarly, we define

$$U_{BF_1} = \{\langle z_0, z_1\rangle \in \mathbb{C}P^1 \,|\, |z_0| < |z_1|\} \times 1,$$

$$U_{CF_1} = \{\langle z_0, z_1\rangle \in \mathbb{C}P^1 \,|\, |z_1| < |z_0|\} \times 1,$$

$$U_{AF_2} = \{\langle z_0, z_1\rangle \in \mathbb{C}P^1 \,|\, |z_0| < |z_1|\} \times 2,$$

$$U_{CF_2} = \{\langle z_0, z_1\rangle \in \mathbb{C}P^1 \,|\, |z_1| < |z_0|\} \times 2.$$

Observe that

$$N_A = N_{AF_2} \oplus N_{AF_0} = L'_{\langle 1,0\rangle} \times 0 \oplus L'_{\langle 0,1\rangle} \times 2,$$

$$N_B = N_{BF_1} \oplus N_{BF_0} = L'_{\langle 0,1\rangle} \times 0 \oplus L'_{\langle 0,1\rangle} \times 1,$$

$$N_C = N_{CF_2} \oplus N_{CF_1} = L'_{\langle 1,0\rangle} \times 1 \oplus L'_{\langle 1,0\rangle} \times 2.$$

We define the maps ψ_{AF_0} in the diagram

$$\begin{tikzcd}
L'_{\langle 1,0\rangle} \times 0 \oplus L'_{\langle 0,1\rangle} \times 2 \arrow[r, "\psi_{AF_0}"] \arrow[d] & \pi^{-1}(U_{AF_0}) \arrow[r, hook] \arrow[d, "\pi"] & L' \times 0 \arrow[d, "\pi"] \\
L'_{\langle 0,1\rangle} \times 2 \arrow[r, "\psi_{AF_0}"] & U_{AF_0} \arrow[r, hook] & F_0
\end{tikzcd}$$

as follows

$$\begin{tikzcd}
(\langle 1,0\rangle,(\lambda_0,0))\times 0\oplus(\langle 0,1\rangle,(0,\lambda_1))\times 2 \arrow[r, "\psi_{AF_0}"] \arrow[d] & \left(\left\langle 1,\frac{\lambda_1}{\sqrt{|\lambda_1|^2+1}}\right\rangle,\left(\frac{\lambda_0\sqrt{|\lambda_1|^2+1}}{\sqrt{2|\lambda_1|^2+1}},\frac{\lambda_0\bar{\lambda}_1}{\sqrt{2|\lambda_1|^2+1}}\right)\right)\times 0 \arrow[d] \\
(\langle 0,1\rangle,(0,\lambda_1))\times 2 \arrow[r, "\psi_{AF_0}"] & \left\langle 1,\frac{\lambda_1}{\sqrt{|\lambda_1|^2+1}}\right\rangle\times 0
\end{tikzcd}$$

The maps ψ_{AF_2} in the diagram

$$\begin{tikzcd}
L'_{\langle 1,0\rangle}\times 0\oplus L'_{\langle 0,1\rangle}\times 2 \arrow[r, "\psi_{AF_2}"] \arrow[d] & \pi^{-1}(U_{AF_2}) \arrow[r, hook] \arrow[d, "\pi"] & L'\times 2 \arrow[d, "\pi"] \\
L'_{\langle 0,1\rangle}\times 0 \arrow[r, "\psi_{AF_2}"] & U_{AF_2} \arrow[r, hook] & F_2
\end{tikzcd}$$

are given by

$$\begin{tikzcd}
(\langle 1,0\rangle,(\lambda_0,0))\times 0\oplus(\langle 0,1\rangle,(0,\lambda_1))\times 2 \arrow[r, "\psi_{AF_2}"] \arrow[d] & \left(\left\langle \frac{\lambda_0}{\sqrt{|\lambda_0|^2+1}},1\right\rangle,\left(\frac{\bar{\lambda}_0\lambda_1}{\sqrt{2|\lambda_0|^2+1}},\frac{\lambda_1\sqrt{|\lambda_0|^2+1}}{\sqrt{2|\lambda_0|^2+1}}\right)\right)\times 2 \arrow[d] \\
(\langle 1,0\rangle,(\lambda_0,0))\times 0 \arrow[r, "\psi_{AF_2}"] & \left\langle \frac{\lambda_0}{\sqrt{|\lambda_0|^2+1}},1\right\rangle\times 2
\end{tikzcd}$$

In a similar manner are defined the maps ψ_{BF_0}, ψ_{BF_1}, ψ_{CF_2} and ψ_{CF_1}. These maps satisfies the hypotheses of our technical theorem in Section 5. Then the manifold M constructed as in the proof of Theorem 2 is

$$M=\frac{L'\times\{0,1,2\}}{\sim},$$

where

$\pi^{-1}(U_{AF_0})\subset L'\times 0$ is identified with $\pi^{-1}(U_{AF_2})\subset L'\times 2$ under the diffeomorphism $\psi_{AF_2}\psi_{AF_0}^{-1}$,

$\pi^{-1}(U_{BF_0})\subset L'\times 0$ is identified with $\pi^{-1}(U_{BF_1})\subset L'\times 1$ under the diffeomorphism $\psi_{BF_1}\psi_{BF_0}^{-1}$, and

$\pi^{-1}(U_{CF_1})\subset L'\times 1$ is identified with $\pi^{-1}(U_{CF_2})\subset L'\times 2$ under the diffeomorphism $\psi_{CF_2}\psi_{CF_1}^{-1}$.

The manifold M is G-equivariantly homeomorphic to the complement of a torus in $\mathbb{C}P^2$. In fact, consider the following open subsets of $\mathbb{C}P^2$:

$$V_0=\{\langle z_0,z_1,z_2\rangle\in\mathbb{C}P^2 \mid |z_0|<\max\{|z_1|,|z_2|\}\},$$

$$V_1=\{\langle z_0,z_1,z_2\rangle\in\mathbb{C}P^2 \mid |z_1|<\max\{|z_0|,|z_2|\}\},$$

$$V_2=\{\langle z_0,z_1,z_2\rangle\in\mathbb{C}P^2 \mid |z_2|<\max\{|z_0|,|z_1|\}\}.$$

Then $V_0 \cup V_1 \cup V_2 = \mathbb{C}P^2 - \mathbf{T}$, where $\mathbf{T}$ is the torus $\{\langle z_0, z_1, z_2\rangle \in \mathbb{C}P^2 \,|\, |z_0| = |z_1| = |z_2|\}$.

On the other hand, we define G-equivariant homeomorphisms $\varphi_i : L' \times i \to V_i$ for $i = 1, 2, 3$ as follows,

$$\varphi_0(\langle z_0, z_1\rangle, (\lambda_0, \lambda_1)) = \langle \varepsilon_j(\langle z_0, z_1\rangle, (\lambda_0, \lambda_1))z_j, z_0, z_1\rangle, \quad \text{if } z_j \neq 0,$$

$$\varphi_1(\langle z_0, z_1\rangle, (\lambda_0, \lambda_1)) = \langle z_0, \varepsilon_j(\langle z_0, z_1\rangle, (\lambda_0, \lambda_1))z_j, z_1\rangle, \quad \text{if } z_j \neq 0,$$

$$\varphi_2(\langle z_0, z_1\rangle, (\lambda_0, \lambda_1)) = \langle z_0, z_1, \varepsilon_j(\langle z_0, z_1\rangle, (\lambda_0, \lambda_1))z_j\rangle, \quad \text{if } z_j \neq 0,$$

where

$$\varepsilon_j(\langle z_0, z_1\rangle, (\lambda_0, \lambda_1)) = \sqrt{\frac{|z_0|^2 + |z_1|^2}{|\lambda_j|^2(|z_0|^2 + |z_1|^2) + |z_j|^2}} \max(|z_0|, |z_1|) \frac{\lambda_j}{|z_j|}$$

$(j = 0, 1)$.

These homeomorphims induce a G-equivariant homeomorphism $\varphi : M \to \mathbb{C}P^2 - \mathbf{T}$, because

in $\pi^{-1}(U_{AF_0})$, the map $\psi_{AF_2}\psi_{AF_0}^{-1}$ coincides with $\varphi_2^{-1}\varphi_0$,

in $\pi^{-1}(U_{BF_0})$, the map $\psi_{BF_1}\psi_{BF_0}^{-1}$ coincides with $\varphi_1^{-1}\varphi_0$,

in $\pi^{-1}(U_{CF_2})$, the map $\psi_{CF_1}\psi_{CF_2}^{-1}$ coincides with $\varphi_1^{-1}\varphi_2$.

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